

HÖLDER CONTINUITY OF ENERGY MINIMIZER MAPS BETWEEN RIEMANNIAN POLYHEDRA

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ABSTRACT.

The goal of the present paper is to establish some kind of regularity of an energy minimizer map between Riemannian polyhedra. More precisely, we will show the hölder continuity of local energy minimizers between Riemannian polyhedra with the target spaces without focal points. With this new result, we also complete our existence theorem obtained in [5], and consequently we generalize completely, to the case of target polyhedra without focal points (which is weaker geometric condition than the nonpositivity of the curvature), the Eells-fuglede's existence and regularity theorem [12, chapters 10, 11] which is the new version of the famous Eells-Sampson's theorem [13].

0. INTRODUCTION.

It is well known that the problems dealing with the existence of energy minimizing maps are related to those of local or global regularity. For example, Eells and Sampson [13] proved that every free homotopy class of maps between smooth Riemannian manifolds, if the target manifolds are of nonpositive sectional curvature, has an energy minimizer which is smooth. Gromov and Schoen [15] extended the Eells-Sampson's results to the case when the target spaces are Riemannian polyhedra and obtained Lipschitz continuous energy minimizers, while Korevaar and Schoen [21] dropped the polyhedral restriction on the target spaces permitting its to be any geodesic spaces. Later Eells and Fuglede [12, chapters 10, 11] proved the existence of hölder continuous energy minimizers between Riemannian polyhedra with the assumption that the target polyhedra are of nonpositive curvature in the sense of Alexandrov [2]. Note that the Riemannian polyhedra are very interesting examples as singular spaces, being harmonic spaces (in the sense of Brelot, see [12] ch 2) and providing a several examples as smooth Riemannian manifolds, triangulable Lipschitz manifolds, Riemannian orbit spaces, singular analytic spaces, stratified spaces, etc...

In our turn, in [5] we expanded the Eells-Fuglede's existence theorem to the case of the target polyhedron without focal points in the sense of [4], but the geometric arguments developed therein did not permit to us to tell something on the local regularity. In fact, we were interested in the Riemannian polyhedra without focal

2000 *Mathematics Subject Classification.* Primary: 58E20, 49N60; Secondary: 58A35.

Key words and phrases. Riemannian Polyhedra, Energy minimizer, Harmonic map, Hölder continuity, Focal point.

points because this class of polyhedra is wider than the class of those of nonpositive Alexandrov's curvature even if the polyhedra are smooth. Indeed, a geodesic space of nonpositive curvature is always without focal points (cf. [4]) while, Gulliver [6] has shown that there are manifolds without focal points of both signs of sectional curvature.

The goal of the present paper is to show the Hölder continuity of local energy minimizers between Riemannian polyhedra with the target spaces without focal points. With this result, we also complete our existence theorem obtained in [5], and consequently, we generalize completely, to the case of target polyhedra without focal points, the last version of the existence and regularity theorem, due to Eells and Fuglede [12, chapters 10, 11]. Remark that, if both the source and the target of the maps are smooth, the theorem is due to Xin (cf. [29]). The methods we will use for solving this problem are (far) different from those used in the smooth case because the aim is also to cover the singular case. Consequently, we will follow the Eells-Fuglede's spirit, but several difficulties arise because of our weaker geometric condition (the absence of the focal points).

The paper is organized as the following. In Section 1, we use the absence of the focal points in a Riemannian polyhedron to produce a strong convexity property of the square of the distance function. We note that establishing this geometric property was quite difficult compared to the Eells-Fuglede's case where the strong convexity of the square of the distance function is a direct consequence of the nonpositivity of the curvature. In section 2, we establish the Hölder continuity of energy minimizer maps between Riemannian polyhedra. Section 3 is devoted to the application of the established regularity to our existence theorem obtained in [5]. To close the paper, and for the sake of completeness, an annex containing an overview of recent metric geometry, Riemannian polyhedra, Energy of map etc..., has been included with references for all the results stated.

Acknowledgements.

I would like to thank Professor J. Eells for bringing to my attention this problem, and for helpful discussions. I owe special thanks to Professor A. Verjovsky for his encouragement and constant support over the years.

1. RIEMANNIAN POLYHEDRA WITHOUT FOCAL POINTS.

This section is devoted to the study of the convexity of the square of the distance function. More precisely we will firstly investigate the case of a simply connected smooth Riemannian manifold without focal points. Secondly, we will use the result obtained in the smooth case to show that the square of the distance function in simply connected Riemannian polyhedra without focal points in sense of [4], is *strongly convex* (see the definition below).

1.1 Smooth Riemannian manifolds without focal points..

The aim of this paragraph is to use the absence of the focal points in a Riemannian manifold to show that the square of the distance function from any fixed point is in some sense strongly convex. Usually, in the literature we talk about the convexity of the square of the distance function and never about its strong convexity. For that reason and for our interests we decide to show this property here.

Let M denote a simply connected complete smooth Riemannian manifold, $t \mapsto \sigma(t)$ a geodesic and $p \in M$ a point not belonging to σ . We have the following proposition.

Proposition 1.1.

If the manifold M is compact without focal points, then the square of the distance function from the point p is strongly convex, that is,
for every geodesic $\sigma : [0, 1] \rightarrow M$, there exists a positive constant a such that:

$$d^2(p, \sigma(s)) \leq (1-s)d^2(p, \sigma(0)) + sd^2(p, \sigma(1)) - as(1-s)d^2(\sigma(0), \sigma(1)).$$

For the proof of the proposition we need the following lemma.

Lemma 1.2.

Let M , σ and p be as in Proposition 1.1. Then there exists a positive constant c such that:

$$\frac{d^2}{ds^2}d^2(p, \sigma(s)) \geq c$$

Proof of Lemma 1.2.

Let us considering a geodesic variation joining the point p and the geodesic σ as follows:

Let $\gamma : (s, t) \in \mathbb{R} \times [0, 1] \mapsto \gamma(s, t)$ be an M -valued map such that, $\gamma(., 1) = \sigma$ and for every fixed s , $\gamma(s, .)$ is minimal geodesic connecting p to $\sigma(s)$.

Set $T := \gamma_* \frac{\partial}{\partial t}$ (the direct image by γ of the vector field $\frac{\partial}{\partial t}$) and $V := \gamma_* \frac{\partial}{\partial s}$. Note that the vector field V is *Jacobi field* [24] with $V(0) = 0$. By a direct computation we obtain:

$$\frac{d^2}{ds^2}d^2(p, \sigma(s)) = \int_0^1 [\langle \nabla_T V, \nabla_T V \rangle - \langle R(T, V)T, V \rangle] dt,$$

where ∇ denote the *symmetric Riemannian connection* of M (relative to its given Riemannian metric) and R is its associate *curvature tensor* [10].

Recall that any such Jacobi field V can be decomposed as follows:

$$V = V_\perp + aT + btT,$$

where V_\perp is a Jacobi field perpendicular to the geodesic $\gamma(s, .)$ and T is the unit tangent vector along $\gamma(s, .)$. Thus we obtain:

$$\frac{d^2}{ds^2}d^2(p, \sigma(s)) = \int_0^1 [\langle \nabla_T V_\perp, \nabla_T V_\perp \rangle - \langle R(T, V_\perp)T, V_\perp \rangle] dt + b^2.$$

The manifold M is assumed without focal points, consequently, on one hand the right term $\int_0^1 [\langle \nabla_T V_\perp, \nabla_T V_\perp \rangle - \langle R(T, V_\perp)T, V_\perp \rangle] dt$ is strictly positive, on the other hand the real number b is nonzero (because $Y = btT$ is a nontrivial Jacobi field vanishing at 0 $|Y(t)|$ is strictly increasing, cf. [26], and M is compact); thus the lemma is thereby proved. \square

Next, we are going to give the proof of Proposition 1.1.

Proof of Proposition 1.1.

It is well known, in the theory of convex functions, that if we are looking for some convexity property on a simply connected Riemannian smooth manifold of a

given function it is enough to show the midconvexity of the relevant property. So, following this tradition, we will show:

for given parametrization of the geodesic σ we have for (closed to 0) $t \geq 0$:

$$d^2(p, \sigma(0)) \leq \frac{1}{2}d^2(p, \sigma(-t)) + \frac{1}{2}d^2(p, \sigma(t)) - at^2.$$

From Lemma 1.2, there is a positive constant c such that $\frac{d^2}{dt^2}d^2(p, \sigma(t))|_{t=0} \geq c$. Then, we obtain for all $\epsilon > 0$ the following inequality:

$$\frac{\frac{d^2(p, \sigma(t)) - d^2(p, \sigma(0))}{t} - \frac{d^2(p, \sigma(0)) - d^2(p, \sigma(-t))}{t}}{t} \geq c - \epsilon,$$

which ends the proof of the proposition. \square

Remarks 1.3.

Proposition 1.1 is also valid if the manifold M is the universal cover of a compact Riemannian manifold without focal points.

1.2 Complete Riemannian polyhedra without focal points.

The goal of this paragraph is to show that in the universal cover (so simply connected) of complete compact Riemannian polyhedron without focal points, the square of the distance function from any fixed point is strongly convex. Recall that in [5], it is already shown that in a simply connected locally compact Riemannian polyhedron the square of the distance function is (just) convex. For simplicity of statements we shall require that, our Riemannian polyhedra are simplexwise smooth. But the results of this paragraph are also valid with mostly the same proofs if the Riemannian polyhedra are just Lip.

Let (X, d_X, g) be a Riemannian polyhedron endowed with simplexwise Riemannian metric g and (K, θ) a fixed triangulation (cf. the annex 3.2).

Recall that for each point $p \in X$ (or $\theta(p) \in K$), there are well defined notions, the tangent cone over p denoted $T_p X$ and the link over p noted $S_p X$ which generalizes respectively the tangent space and the unit tangent space if X is also smooth manifold (see the annex 3.1).

Now, we state the main result of this section.

Theorem 1.4.

Assume that the polyhedron X is compact and without focal points. Let \tilde{X} denote its universal cover and p a point of \tilde{X} . Then for every geodesic $\sigma : I \subseteq \mathbb{R} \rightarrow \tilde{X}$, the square of the distance function from the point p to the geodesic σ is strongly convex, that is,

there exists a positive constant c depending only on the polyhedron X such that:

$$d^2(p, \sigma(s)) \leq (1-s)d^2(p, \sigma(0)) + sd^2(p, \sigma(1)) - cs(1-s)d^2(\sigma(0), \sigma(1)).$$

Before giving the proof of Theorem 1.4, let us start by the following remarks:

Remarks 1.5.

- (1) *Alexander and Bishop [1] have shown that a simply connected complete locally convex geodesic space is globally convex. Thus, following the same argument than the one used by Alexander-Bishop, to prove Theorem 1.4, it is only required to show that every point $x \in \tilde{X}$ admits an open convex neighborhood U_x . In other terms, we just need to show the following: For every $x \in \tilde{X}$ there is an open neighborhood U_x such that, every geodesic σ with end points in U_x belongs to U_x and the function $L : t \mapsto d^2(x, \sigma(t))$ is strongly convex.*
- (2) *It is shown in [5] that if Y is complete simply connected Riemannian polyhedron without focal points then, for every geodesic $\sigma \subset Y$, the function $L : t \mapsto d^2(x, \sigma(t))$ is continuous and convex.*

Proof of Theorem 1.4.

Let (X, g, d) be a compact polyhedron, $(\tilde{X}, \tilde{g}, \tilde{d})$ its universal cover (so it is complete and simply connected) and (K, θ) a triangulation of \tilde{X} . The fundamental group $\pi_1(X)$ acts isometrically and simplicially on \tilde{X} thus there exists a compact set $\tilde{F} \subset \tilde{X}$ called a *fundamental domain* of $\pi_1(X)$ whose boundary $\partial\tilde{F}$ has measure 0 and each point of \tilde{X} is $\pi_1(X)$ -equivalent either to exactly one point of the interior of \tilde{F} or to at least one point of $\partial\tilde{F}$. The fact that X is compact implies that the compact \tilde{F} can be obtained as a suitable finite union of maximal simplexes of \tilde{X} .

In the following we will omit the homeomorphism of the (some) triangulation in our notations and so we will not do any distinction between the simplexes of \tilde{X} and the simplexes of K . By the first remark of 1.5 it is only required to show the locale strong convexity (in our sense) and for points belonging to the fundamental domain. So there are two cases to investigate, the first one is when the point p is in the topological interior of some maximal simplex of the fundamental domain \tilde{F} and the second one is when the point $p \in \tilde{F}$ is vertex (to the triangulation (K, θ)).

Suppose that p is in the interior of the maximal simplex Δ . Then there exists a positif reel $r_p > 0$ such that the open ball $B(p, r_p)$ with center p and ray r_p is contained in Δ . Thanks to the Riemannian metric g_Δ , the open Ball $B(p, r_p)$ can be thought of as sub-manifold of some compact simply connected smooth Riemannian manifold endowed with the Riemannian metric g_Δ . Take now a geodesic σ with end points in $B(p, r_p)$ then by the second remark of 1.5 it is contained in the ball $B(p, r_p)$. The polyhedron \tilde{X} is without focal points so the neighborhood (sub-manifold) $B(p, r_p)$ is without focal points too. Thus, by Proposition 1.1, the function L is strongly convex for every geodesic σ contained in $B(p, r_p)$.

Now, look at the case when p is vertex of \tilde{F} . Let r_p be a positif reel such that the open ball $B(p, r_p)$ is included in the open star $st(p)$ of p (see, Annex 3.1). Let $\sigma : [a, b] \rightarrow \tilde{F}$ be a geodesic of $B(p, r_p)$ and let $\bigcup_i \Delta_i^o$ (finite union because is locally compact) denote the star of p . We know that there is a subdivision $t_0 = a, t_1, \dots, t_n = b$ such that each restriction $\sigma|_{[t_i, t_{i+1}]} \subset \Delta_i$ is a geodesic in sense of smooth Riemannian geometry. So by Proposition 1.1 and the second remark of 1.5, the question about the strong convexity of the function $L(t) = d^2(p, \sigma(t))$ is asked when σ transits from a simplex Δ_i to a simplex Δ_{i+1} i.e. at the points t_i .

Reparameterizing the geodesic σ and suppose that for fixed j , $t_j = 0$ and that for small $\epsilon > 0$ the geodesic segment $\sigma|_{[-\epsilon, 0]}$ is included in the maximal simplex Δ_1

and $\sigma|_{[0,\epsilon]}$ is included in the maximal simplex Δ_2 (by taking ϵ small enough).

By Remark 1.5 the function $L(t) = d^2(p, \sigma(t))$ is continuous then, as it is used in the theory of convex functions, it is only required to show the strong midconvexity of the function L i.e.

$$\text{for every } t \in [0, \epsilon], L(0) \leq \frac{1}{2}L(-t) + \frac{1}{2}L(t) - ct^2.$$

Every maximal simplex Δ_i from the star of p is thought of as a cell of some smooth compact simply connected Riemannian manifold (M_i, g_i) without focal points. Recall that there is an exponential function (diffeomorphism) defined from the tangent bundle TM_i of each manifold M_i to M_i . Let us now consider the two exponential maps $\exp_1 : T_p M_1 \rightarrow M_1$ and $\exp_2 : T_p M_2 \rightarrow M_2$. The spaces $T_i M$, $i = 1, 2$ can be assimilated to an euclidean space of dimension lower or equal the dimension of the polyhedron X . Consequently, there is an isometry $I : T_p M_1 \rightarrow T_p M_2$. Then, thanks to the diffeomorphism $\exp_2 \circ I \circ \exp_1^{-1}$ (and its inverse), we can compare (point by point) the associate distance functions d_1, d_2 to the Riemannian metrics g_1, g_2 in the following sense:

$$\text{for every point } q \in M_1, \text{ compare } d_1(p, q) \text{ and } d_2(p, \exp_2 \circ I \circ \exp_1^{-1}(q)).$$

With this possibility, comparing the distance functions d_1 and d_2 , we can suppose for example that $d_1 \leq d_2$ for some $t \in [-\epsilon, \epsilon]$.

By the definition of the distance function \tilde{d} of the polyhedron \tilde{X} , we have: $\tilde{d}(p, \sigma(0)) \leq d_1(p, \sigma(0))$. Now consider the concatenation $\sigma_1 \subset M_1$ of the two geodesics $\sigma|_{[-t, 0]}$ and $\exp_1 \circ I^{-1} \circ \exp_2^{-1}(\sigma|_{[0, t]})$. Then by the fact that the polyhedron \tilde{X} is without focal points and the distance comparison hypothesis, σ_1 is minimal geodesic and the function $t \mapsto d_1^2(p, \sigma_1)$ is strongly convex i.e.

$$\tilde{d}^2(p, \sigma(0)) \leq d_1^2(p, \sigma(0)) \leq \frac{1}{2}d_1^2(p, \sigma(-t)) + \frac{1}{2}d_1^2(p, \exp_1 \circ I^{-1} \circ \exp_2^{-1}(\sigma(t))) - c_1 t^2.$$

But we have supposed that $d_1 \leq d_2$ at t so $d_1^2(p, \exp_1 \circ I^{-1} \circ \exp_2^{-1}(\sigma(t))) \leq d_2^2(p, \sigma(t))$ which leads to:

$$\tilde{d}^2(p, \sigma(0)) \leq d_1^2(p, \sigma(0)) \leq \frac{1}{2}d_1^2(p, \sigma(-t)) + \frac{1}{2}d_2^2(p, \sigma(t)) - c_1 t^2.$$

Now if we take $c = \inf(c_1, c_2)$ we obtain:

$$\tilde{d}^2(p, \sigma(0)) \leq \frac{1}{2}d_1^2(p, \sigma(-t)) + \frac{1}{2}d_2^2(p, \sigma(t)) - ct^2. (*)$$

But in the interior of each Δ_i , we have $d_i = \tilde{d}$ (may be we should to take an r_p smaller) so we have:

$$\tilde{d}^2(p, \sigma(0)) \leq \frac{1}{2}\tilde{d}^2(p, \sigma(-t)) + \frac{1}{2}\tilde{d}^2(p, \sigma(t)) - ct^2.$$

For ending the proof, just remark firstly that for every $t \in [-\epsilon, \epsilon]$, the two distance functions d_1, d_2 are comparable (in the above sense). Secondly the inequality $(*)$ is

symmetric in d_1 and d_2 (because if $d_2 \leq d_1$ just inverse the role of d_1 and d_2 in the proof and we obtain the same inequality). Thirdly we can choose the c uniformly because \tilde{X} is locally compact and \tilde{F} is compact. Finally the polyhedron \tilde{X} is simply connected so by Remarks 1.5 (the first remark) the local strong convexity of the square of the distance function becomes global for a constant c depending only on the fundamental domain \tilde{F} and consequently it depends only on the polyhedron X . \square

Before ending this section, we mention the following remarks:

Remarks 1.6.

- (1) *Theorem 1.4 is also valid if the space \tilde{X} is compact simply connected Riemannian polyhedron without focal points (not necessary the universal cover of a compact Riemannian polyhedron), or it is a Riemannian polyhedron with bounded geometry in sense of [6].*
- (2) *When the polyhedron X is complete of nonpositive curvature (in the sense of Alexandrov) and not necessary compact, the constant c in Theorem 1.4 is equal to 1.*

2. HÖLDER CONTINUITY

In the present section we will discuss a kind of regularity of an energy minimizer map between Riemannian polyhedra. More precisely we will take a two Riemannian polyhedra (X, g) and (Y, h) of dimensions m and n , with X admissible and simplexwise smooth, and Y compact and without focal points ; and we ask the question: what level of regularity of a given locally energy minimizing map $\varphi : X \rightarrow \tilde{Y}$ (\tilde{Y} is the universal cover of Y) can we have? The best answer we obtain is the following theorem.

Theorem 2.1.

Let X and Y be Riemannian polyhedra. Suppose that X is admissible and simplexwise smooth and Y is compact without focal points. Then, every locally energy minimizing map $\varphi : X \rightarrow \tilde{Y}$, where \tilde{Y} is the universal cover of Y , is Hölder continuous.

Recall that in our context, a map $\varphi : X \rightarrow \tilde{Y}$ is Hölder continuous if there is a Hölder continuous map which is equals to φ almost everywhere in X .

To prove Theorem 2.1, we will adapt to our frame the arguments used by Eells and Fuglede in [12] where they proved a similar theorem but in the case of the target polyhedron a complete Riemannian polyhedron of nonpositive curvature. However, the difficulties in our case come firstly, from the fact that we are considering the Riemannian polyhedra as geometric habitat and where, in general, we can not use the second differential calculus. Secondly, difficulties also arise from the fact that the strong convexity of the square of the distance function established in Section 1 is quite weaker than the convexity property used by Eells-Fuglede in [12] and it is also optimal in our case.

Now let us begin with some lemmas which will be needed for the proof of Theorem 2.1.

Lemma 2.2.

Let Y be a compact Riemannian polyhedron without focal points. Let $\varphi \in W_{loc}^{1,2}(X, \tilde{Y})$ be locally energy minimizing map, where $(\tilde{Y}, d_{\tilde{Y}})$ is the universal cover of Y . Then φ is essentially locally bounded (i.e. $\varphi = \tilde{\varphi}$, a.e. and $\tilde{\varphi}$ is bounded) and we have for any $q \in \tilde{Y}$:

- (1) The functions $v_q : x \mapsto d_{\tilde{Y}}(q, \varphi(x))$ and $v_q^2 : x \mapsto d_{\tilde{Y}}^2(q, \varphi(x))$ from $W_{loc}^{1,2}(X)$ are weakly subharmonic.
- (2) $E(v_q^2, \lambda) \leq -2\frac{1}{c} \int_X e(\varphi) \lambda d\mu_g$ for every $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$ which we can write in weak sense: $\Delta v_q^2 \geq 2\frac{1}{c} e(\varphi)$, with c is the constant of the strong convexity of the square of the distance function, where $E(v_q^2, \lambda) := \int_X c_m \langle \nabla \lambda, \nabla v_q^2 \rangle d\mu_g$ with $c_m = \frac{\omega_m}{m+1}$ and, ω_m being the volume of the unit ball in \mathbb{R}^m and ∇ and $\langle \cdot, \cdot \rangle$ denote respectively the gradient operator and the inner product, defined a.e in X .

Proof Lemma 2.2.

Following the same idea used by Eells-Fugled (Lemma 10.2 in [12]), an idea used by Jost [19] and before used in [20], we will compare the map φ as in the Lemma 2.2 with maps obtained by pulling $\varphi(x)$ towards a given point $q \in \tilde{Y}$.

The map $\varphi : X \rightarrow \tilde{Y}$ from the space $W_{loc}^{1,2}(X, \tilde{Y})$ is locally energy minimizing, so X can be covered by relatively compact domains $U \subset X$ for which $E(\varphi|_U) \leq E(\psi|_U)$ for every map $\psi \in W_{loc}^{1,2}(X, \tilde{Y})$ such that $\psi = \varphi$ a.e. in $X \setminus U$.

Note $v = v_q : x \mapsto d_{\tilde{Y}}(q, \varphi(x))$, $v^2 = v_q^2 : x \mapsto d_{\tilde{Y}}^2(q, \varphi(x))$ and set d refereing to the distance function $d_{\tilde{Y}}$ in \tilde{Y} . All the properties we want to show are local so we can suppose that X is compact and it does not alter the results of the lemma.

By Theorem 1.4 the square of the distance function is strongly convex so there exists a constant $c > 0$ (depending only on the polyhedron Y) such that for every geodesic σ arc length parameterized we have:

$$d^2(p, \sigma(s)) \leq (1-s)d^2(p, \sigma(0)) + sd^2(p, \sigma(1)) - cs(1-s)d^2(\sigma(0), \sigma(1)).$$

There are two case to investigate, the first one is when $c \geq 1$ and the second one is when $c < 1$.

In the first case $c \geq 1$, the strong convexity of the square distance function implies the following:

$$d^2(p, \sigma(s)) \leq (1-s)d^2(p, \sigma(0)) + sd^2(p, \sigma(1)) - s(1-s)d^2(\sigma(0), \sigma(1)),$$

which join the case of nonpositive curvature and so by Eells-fuglede's results ([12] ch 10) the Lemma 2.2 follows.

Now look at the second case when $0 < c < 1$. Let us first consider the case when λ is lipschitz map on X , $0 \leq \lambda \leq 1$ and the support of λ noted $supp \lambda$ is subset of some domain U as we mentioned in the beginning of the proof. Let γ_x denote the minimal geodesic in the space \tilde{Y} joining the point $\gamma_x(0) = \varphi(x)$ to $\gamma_x(1) = q$ for $x \in X$. Define a map (the pulling of φ) $\varphi_\lambda : X \rightarrow \tilde{Y}$ by $\varphi_\lambda(x) = \gamma_x(\lambda(x))$, for $x \in X$.

The function φ_λ is $L^2(X, \tilde{Y})$ because the geodesic γ_x varies continuously with its end point $\varphi(x)$ (The space \tilde{X} is without conjugate points, see [5]) and $d(\varphi_\lambda(x), q) \leq d(\varphi(x), q)$ with $d(\varphi(\cdot), q) \in L^2(X)$ (since $\varphi \in L^2(X, \tilde{Y})$).

The strong convexity of the square of the distance function gives, for $x, x' \in X$,

$$\begin{aligned} d^2(\varphi(x), \varphi_\lambda(x')) &\leq (1 - \lambda(x'))d^2(\varphi(x), \varphi(x')) + \lambda(x')d^2(\varphi(x), q) - \\ &\quad c\lambda(x')(1 - \lambda(x'))d^2(\varphi(x'), q) \end{aligned}$$

and

$$d^2(\varphi_\lambda(x), \varphi_\lambda(x')) \leq (1 - \lambda(x))d^2(\varphi(x), \varphi_\lambda(x')) + \lambda(x)d^2(\varphi_\lambda(x'), q) - \\ c\lambda(x)(1 - \lambda(x))d^2(\varphi(x), q).$$

Combining the two inequalities and inserting $d(\varphi_\lambda(x'), q) = (1 - \lambda(x'))d(\varphi(x'), q)$, we obtain,

$$d_\lambda^2 - d^2 \leq -[\lambda(x) + \lambda(x') - \lambda(x)\lambda(x')]d^2 + (1 - \lambda(x))\lambda(x')v^2(x) - c\lambda(x)(1 - \lambda(x))v^2(x) \\ - c(1 - \lambda(x))\lambda(x')(1 - \lambda(x'))v^2(x') + \lambda(x)(1 - \lambda(x'))^2v^2(x'),$$

with $d_\lambda = d(\varphi_\lambda(x), \varphi_\lambda(x'))$ and $d = d(\varphi(x), \varphi(x'))$.

Now, we will pull closely the map φ towards the point q such that $\lambda \geq \frac{1}{1+c}$. Under this assumption ($\lambda \geq \frac{1}{1+c}$), we have for every $x, x' \in X$, $\lambda(x)(1 - \lambda(x')) \leq c\lambda(x)(1 - c\lambda(x'))$ and $\lambda(x')(1 - \lambda(x)) \leq c\lambda(x')(1 - c\lambda(x))$. Taking in account these inequalities we obtain,

$$d_\lambda^2 - d^2 \leq -[\lambda(x) + \lambda(x') - \lambda(x)\lambda(x')]d^2 - c(\lambda(x) - \lambda(x'))(v^2(x) - v^2(x')) \\ + |O(\lambda^2(x))| + |O(\lambda(x')\lambda(x))| + |O(\lambda(x)\lambda^2(x'))|.$$

For given λ a compact supporting positive lipschitz function, replace λ with $t\lambda$, for any $0 < t < \epsilon$, this leads to,

$$d_{t\lambda}^2 - d^2 \leq -[t\lambda(x) + t\lambda(x') - t^2\lambda(x)\lambda(x')]d^2 - ct(\lambda(x) - \lambda(x'))(v^2(x) - v^2(x')) \\ + \epsilon^2|O(\lambda^2(x))| + \epsilon^2|O(\lambda(x')\lambda(x))| + \epsilon^3|O(\lambda(x)\lambda^2(x'))| \quad (*).$$

Observe that

$$E(\varphi_{t\lambda|U}) - E(\varphi|U) = \lim_{f \in C_c(U, [0,1])} \limsup_{\epsilon \rightarrow 0} \int_U (e_\epsilon(\varphi_{t\lambda}) - e_\epsilon(\varphi)) f d\mu_g,$$

where for $\epsilon > 0$, $e_\epsilon(\psi)(x) = \int_{B_X(x, \epsilon)} \frac{d_Y^2(\psi(x), \psi(x'))}{\epsilon^{m+2}} d\mu_g(x')$, with $\psi \in L_{loc}^2(X, \tilde{Y})$.

Note that for $x' \in B_X(x, \epsilon)$ we have $|\lambda(x) - \lambda(x')| \leq \text{const} \cdot \epsilon$ and φ is supposed locally energy minimizing, so by Eells-Fuglede's results [12, Definition 9.1, Theorem 9.1 and Corollary 9.2] we deduce from the inequality (*),

$$0 \leq E(\varphi_{t\lambda|U}) - E(\varphi|U) \leq - \int_U (2t\lambda - t^2\lambda^2) e(\varphi) d\mu_g - cc_m \int_U t \langle \nabla \lambda, \nabla v^2 \rangle d\mu_g + \\ \int_U c_m (O(\|\lambda\|_{L^\infty}^2) + O(\|\lambda\|_{L^\infty}^3)) d\mu_g,$$

where c_m is constant depending on the polyhedron X and, ∇ and $\langle \cdot, \cdot \rangle$ denote respectively the gradient operator and the inner product, defined a.e in X (cf. [21] and [12, ch 5]). Thus the function $\varphi_{t\lambda}$ is in the space $W_{loc}^{1,2}(X)$.

Now again replace in the last inequalities λ with $t^2\lambda$ divided by t^3 and let $t \rightarrow 0$ we obtain for every $\lambda \in Lip_c^+(U)$,

$$0 \leq - \int_U 2\lambda e(\varphi) d\mu_g - cc_m \int_U \langle \nabla \lambda, \nabla v^2 \rangle d\mu_g.$$

So we infer that for every $\lambda \in Lip_c^+(U)$,

$$0 \leq \int_U 2\lambda e(\varphi) d\mu_g \leq -cc_m \int_U \langle \nabla \lambda, \nabla v^2 \rangle d\mu_g.$$

These inequalities extend to a functions $\lambda \in W_c^{1,2}(U) \cap L^\infty(U)$, with $\lambda \geq 0$, because any such λ can be approximated in $W_c^{1,2}(U)$ by uniformly bounded functions in $L_c^+(U)$. Thus for $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$ we have (on X),

$$0 \leq \int_X 2\lambda e(\varphi) d\mu_g \leq -cc_m \int_X \langle \nabla \lambda, \nabla v^2 \rangle d\mu_g.$$

So we have shown the second statement of Lemma 2.2. For the first part of the lemma, just remark that by the last inequalities we have, for every $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$,

$$\int_X \langle \nabla \lambda, \nabla v^2 \rangle d\mu_g \leq 0,$$

which means that the function $v^2 = d^2(\varphi(\cdot), q)$ is weakly subharmonic in X , and in particular essentially locally bounded.

For the function $v = d(\varphi(\cdot), q)$, by the usual polarization [12, page 21 (2.1)] we have for every $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$,

$$E(v^2, \lambda) = 2E(v, \lambda v) - 2c_m \int_X \lambda \langle \nabla v, \nabla v \rangle d\mu_g.$$

Remember that by the triangle inequality, $|v(x) - v(x')|^2 \leq d^2(\varphi(x), \varphi(x'))$, and so [12, corollary 9.2] $c_m |\nabla v|^2 \leq e(\varphi)$. Inserting $E(v^2, \lambda) \leq -\frac{1}{c} \int_X 2\lambda e(\varphi) d\mu_g \leq 0$ in the last equality, it therefor follows, for every $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$,

$$E(v, \lambda v) = \int_X c_m \langle \nabla v, \nabla(\lambda v) \rangle d\mu_g \leq (1 - \frac{1}{c}) \int_X \lambda e(\varphi) d\mu_g.$$

But the constant c is supposed < 1 so we deduce that,

$$\text{for every } \lambda \in W_c^{1,2}(X) \cap L^\infty(X), \lambda \geq 0, \int_X \langle \nabla v, \nabla(\lambda v) \rangle d\mu_g \leq 0.$$

Now, using the Eells-Fuglede's arguments [12, page 183], we deduce that the function v is weakly subharmonic in X and it is essentially locally bounded too. \square

Corollary 2.3.

Under the hypotheses of Lemma 2.2, if the map φ is of (global) finite energy then, $E(v^2, \lambda) \leq -2\frac{1}{c} \int_X e(\varphi) \lambda d\mu_g$ for every $\lambda \in W_0^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$.

Proof of Corollary 2.3.

By Lemma 2.2, we have, for every $\lambda \in W_c^{1,2}(X) \cap L^\infty(X)$, $\lambda \geq 0$, $E(v^2, \lambda) \leq -2\frac{1}{c} \int_X e(\varphi) \lambda d\mu_g$.

By truncation, any positive function $\lambda \in W_0^{1,2}(X) \cap L^\infty(X)$ can be approximated in $W_0^{1,2}(X)$ by a uniformly bounded sequence of function $\lambda_n \in W_c^{1,2}(X) \cap L^\infty(X)$.

Now, $E(\varphi)$ is supposed $< \infty$, so by the dominated convergence theorem, we have $\int \lambda_n e(\varphi) d\mu_g \rightarrow \int \lambda e(\varphi) d\mu_g$ (in fact there is a subsequence of (λ_n) which converges to λ pointwise a.e.)

□

Corollary 2.4.

Under the hypotheses of Lemma 2.2, every locally energy minimizing map $\varphi : X \rightarrow \tilde{Y}$ is locally essentially bounded.

Proof of Corollary 2.4.

The Eells-fuglede's proof [12, Corollary 10.1] of the same statement in the case where the target space Y is of nonpositive curvature, remains valid in our case.

□

The following lemma will be necessary in the proof of the next one, but it is also of special self interest.

Lemma 2.5.

Let (\mathfrak{N}, ν) be a probability measure space, let (\tilde{Y}, d) be the universal covering of compact Riemannian polyhedron Y without focal points, and $f \in L^2(\mathfrak{N}, \tilde{Y})$. Then there exists a unique center of mass \bar{f}_ν , defined as the point in \tilde{Y} which minimizes the integral $\int_{\mathfrak{N}} d^2(f(x), q) d\nu(x)$.

Proof of Lemma 2.5.

The space \tilde{Y} is supposed without focal points, consequently the square of the distance function is strong convex. So, if y_1, y_2 are two points in \tilde{Y} and $y_{\frac{1}{2}}$ is their midpoint (the unique point in the unique geodesic between y_1 and y_2 which is at equal distance to both y_1 and y_2), then we have,

$$d^2(f(x), y_{\frac{1}{2}}) \leq \frac{1}{2} d^2(f(x), y_1) + \frac{1}{2} d^2(f(x), y_2) - \frac{1}{4} c d^2(y_1, y_2),$$

with $0 < c$ a constant depending on the space Y . Integrating over \mathfrak{N} we obtain,

$$\frac{1}{4} c d^2(y_1, y_2) \leq \frac{1}{2} \int_{\mathfrak{N}} d^2(f(x), y_1) d\nu(x) + \frac{1}{2} \int_{\mathfrak{N}} d^2(f(x), y_2) d\nu(x) - \int_{\mathfrak{N}} d^2(f(x), y_{\frac{1}{2}}) d\nu(x),$$

Thus any minimizing sequence (x_i) is Cauchy, in particular it converges to a unique limit point (\tilde{Y} is complete) which is the unique minimizer of our integral.

□

Lemma 2.6.

Let (X, g) denote a compact admissible Riemannian polyhedron and (\tilde{Y}, d) be the universal covering of compact Riemannian polyhedron Y without focal points. For every measurable set $A \subset X$ with $\mu_g(A) > 0$, the meanvalue $\bar{\varphi}_A \in \tilde{Y}$ over A of a map $\varphi \in W^{1,2}(X, Y)$, defined as the minimizing point in \tilde{Y} of the integral $\int_A d^2(\varphi(x), q) d\mu_g(x)$, lies in the closed convex hull of the essential image $\varphi(A)$.

Before proving the lemma, just recall that the *essential image* $\varphi(A)$ is defined as the closed set of all points $q \in \tilde{Y}$ such that $A \cap \varphi^{-1}(V)$ has positive measure for any neighborhood V of q in \tilde{Y} , and the *closed convex hull* of a set $B \subset \tilde{Y}$ is defined as the intersection of all closed convex subsets of \tilde{Y} containing B .

Proof of Lemma 2.6.

The existence and uniqueness of the meanvalue point (of a map belonging to $W^{1,2}(X, Y)$) over any measurable subset of X are immediately deduced from Lemma 2.5.

Let C denote any convex set containing $\varphi(A)$, let $y \in \tilde{Y} \setminus C$, we claim that there is unique point $\hat{y} \in C$ nearest to y . Indeed, any minimizing sequence $(y_i) \subset C$ for the function $d^2(y, \cdot)$ on C , is Cauchy in \tilde{Y} , by the strong convexity of the square of the distance function, and hence has a unique limit point $\hat{y} \in C$ (because C is closed).

Now, let z denote any point of C , consider the unique geodesic $\sigma_{z\hat{y}}$ (because \tilde{Y} is without conjugate points [5]) connecting z to \hat{y} . The point \hat{y} is the unique orthogonal projection (in sense of [5]) of the point y on the geodesic $\sigma_{z\hat{y}}$, consequently the angle at the point \hat{y} (the distance in the link of \hat{y}) between the geodesics $\sigma_{z\hat{y}}$ and $\sigma_{\hat{y}y}$ is $\geq \frac{\pi}{2}$, and so, we deduce that, $d(z, y) > d(z, \hat{y})$.

Moreover, we infer that $d(\varphi(\cdot), y) > d(\varphi(\cdot), \hat{y})$ a.e in A , which rules out the possibility that any point $y \in \tilde{Y} \setminus C$ can be the meanvalue of φ over A . □

Now, we have all the ingredients to prove Theorem 2.1.

Proof of Theorem 2.1.

Replacing Lemma 10.2, Lemma 10.4 and Corollary 10.1 in the Eells-Fuglede's proof of the equivalent theorem [12, Ch10, page 189] in the case of the target polyhedron of nonpositive curvature, with our Lemma 2.2, Lemma 2.6 and Corollary 2.4 respectively, using the weak Poincaré inequality [cf. 12, Proposition 9.1] and the fact that Sublemma 10.1, Lemma 10.3 and Corollary 10.2 of [12] remain valid in our case, then using the Eells-Fuglede's arguments [12, pages 189-192], we easily derive our theorem. □

3. APPLICATION

A naturel question comes to our minds after the regularity result of Theorem 2.1: is when or where can we apply the regularity obtained? Thus, we will close the paper with this short section where we will give an example of such application. The application proposed will in some sense, complete the existence result of energy minimizer maps, obtained in [5]. Henceforth all polyhedra considered are supposed simplexwise smooth.

Theorem 3.1.

Let X and Y be compact Riemannian polyhedra. Suppose that X is admissible and Y is without focal points.

Then every homotopy class $[u]$ of each continuous map u between the polyhedra X and Y has an energy minimizer relative to $[u]$ which is Hölder continuous.

Proof of Theorem 3.1.

Let X and Y be two compact Riemannian polyhedra such that X is admissible and Y is without focal points.

Firstly, remark that the existence part of Theorem 3.1 is already proved in [5].

Secondly, as we showed in the proof of the existence part in [5], if u denote an energy minimizer in the class $[u]$ then it can be covered by a map $\tilde{u} : \tilde{X} \rightarrow \tilde{Y}$, where \tilde{X} and \tilde{Y} denote respectively the universal covers of X and Y , and which minimizes the energy in the class of the equivariant maps with respect to the fundamental groups $\pi_1(X)$ and $\pi_1(Y)$ in $W^{1,2}(\tilde{X}, \tilde{Y})$. Moreover $E(u) = \int_{\tilde{F}} e(\tilde{u})$, where $\tilde{F} \subset \tilde{X}$ denote the fundamental domain of $\pi_1(X)$. But the universal cover \tilde{Y} satisfies the hypothesis of Theorem 2.1 so \tilde{u} is hölder continuous, and therefore is the map u (the energy minimizer map relative to the class $[u]$). This ends the proof. \square

ANNEX.

The annex is globally devoted to an overview concerning the geodesic spaces, Riemannian polyhedra and the harmonic maps on singular spaces. The last subject was developed successively by Gromov-Schoen [15], Korevaar-Schoen [21] [22] and Eells-Fuglede [12]. We hope that the annex will be useful for the reader.

1. Geodesic spaces [2] [6] [7] [8] [14].

Let X be a metric space with metric d . A curve $c : I \rightarrow X$ is called a *geodesic* if there is $v \geq 0$, called the speed, such that every $t \in I$ has neighborhood $U \subset I$ with $d(c(t_1), c(t_2)) = v|t_1 - t_2|$ for all $t_1, t_2 \in U$. If the above equality holds for all $t_1, t_2 \in I$, then c is called *minimal geodesic*.

The space X is called a *geodesic space* if every two points in X are connected by minimal geodesic. We assume from now on that X is complete geodesic space.

A triangle Δ in X is a triple $(\sigma_1, \sigma_2, \sigma_3)$ of geodesic segments whose end points match in the usual way. Denote by H_k the simply connected complete surface of constant Gauss curvature k . A *comparison triangle* $\bar{\Delta}$ for a triangle $\Delta \subset X$ is a triangle in H_k with the same lengths of sides as Δ . A comparison triangle in H_k exists and is unique up to congruence if the lengths of sides of Δ satisfy the triangle inequality and, in the case $k > 0$, if the perimeter of Δ is $< \frac{2\pi}{\sqrt{k}}$. Let $\bar{\Delta} = (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3)$ be a comparison triangle for $\Delta = (\sigma_1, \sigma_2, \sigma_3)$, then for every point $x \in \sigma_i$, $i = 1, 2, 3$, we denote by \bar{x} the unique point on $\bar{\sigma}_i$ which lies at the same distances to the ends as x .

Let d denote the distance functions in both X and H_k . A triangle Δ in X is *CAT_k triangle* if the sides satisfy the triangle inequality, the perimeter of Δ is $< \frac{2\pi}{\sqrt{k}}$ for $k > 0$, and if $d(x, y) \leq d(\bar{x}, \bar{y})$, for every two points $x, y \in X$.

We say that X has curvature at most k and write $k_X \leq k$ if every point $x \in X$ has a neighborhood U such that any triangle in X with vertices in U and minimizing sides is CAT_k. Note that we do not define k_X . If X is Riemannian manifold, then $k_X \leq k$ iff k is an upper bound for the sectional curvature of X .

A geodesic space X is called geodesically complete iff every geodesic can be stretched in the two direction.

We say that a geodesic space X is without conjugate points if every two points in X are connected by unique geodesic.

2. Orthogonality and focal point.

For more details on the study of focal points in geodesic space, the reader can see [4] and [5].

2.1 Orthogonality.

(X, d) will denote a complete geodesic space. Let $\sigma : \mathbb{R} \rightarrow X$ denote a geodesic and $\sigma_1 : [a, b] \rightarrow X$ a minimal geodesic with a foot in σ (i.e. $\sigma_1(a) \in \sigma(\mathbb{R})$).

The geodesic σ_1 is *orthogonal* to σ if for all $t \in [a, b]$, the point $\sigma_1(t)$ is locally of minimal distance from σ .

In the case when for given geodesic σ and a non-belonging point p there exists an orthogonal geodesic σ' to σ and containing p , we will call the intersection point between σ and σ' the *orthogonal projection point* of p on σ .

It is shown in [4] that, on the one hand, if the geodesic σ is minimal then there always exists a realizing distance orthogonal geodesic to σ connecting every external point p (off σ) to σ . On the other hand, If the space (X, d) is locally compact with non-null injectivity radius and the geodesic σ is minimal on every open interval with length lower than the injectivity radius, then for every point p off σ and whose distance from σ is not greater than the half of the injectivity radius, there exists a geodesic joining orthogonally the point p and the geodesic σ .

As corollaries, if the space (X, d) is simply connected CAT_0 space then for given geodesic $\sigma : \mathbb{R} \rightarrow X$ and an off point p there always exists a realizing distance orthogonal geodesic from p to σ . When X is CAT_k for positive constant k then there always exists an orthogonal geodesic to σ from a point p whose distance from σ is not greater than $\frac{\pi}{2\sqrt{k}}$. In these last two cases the angle between two orthogonal geodesics (in the sense of the definition above) is always greater than or equal to $\frac{\pi}{2}$.

2.2 Focal points.

Let (X, d) denote a complete geodesic space, $\sigma : \mathbb{R} \rightarrow X$ a geodesic and p a point not belonging to the geodesic σ .

The point p is said a *focal point* of the geodesic σ or just a focal point of the space X , if there exists a minimal geodesic variation $\tilde{\sigma} :]-\epsilon, \epsilon[\times [0, l] \rightarrow X$ such that, if we note $\tilde{\sigma}(t, s) = \sigma_t(s)$, σ_0 is minimal geodesic joining p to the point $q = \sigma(0)$ and for every $t \in]-\epsilon, \epsilon[$, σ_t is minimal geodesic containing $\sigma(t)$, with the properties:

- (1) For every $t \in]-\epsilon, \epsilon[$, each a geodesic σ_t is orthogonal to σ .
- (2) $\lim_{t \rightarrow 0} \frac{d(p, \sigma_t(l))}{d(q, \sigma(t))} = 0$.

This definition was introduced in [4], as a natural generalization of the same notion in the smooth case. It is shown in the same paper that the Hadamard spaces are without a focal point.

It is also shown in [5], that if X is simply connected geodesic space without focal points then it is without conjugate points.

3. Riemannian polyhedra.

3.1 Riemannian admissible complexes ([3] [6] [7] [11] [28]).

Let K be locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric g ; i.e. g is a family of smooth Riemannian metrics g_Δ on simplices Δ of K such that the restriction $g_\Delta|_{\Delta'} = g_{\Delta'}$ for any simplices Δ' and Δ with $\Delta' \subset \Delta$.

Let K a finite dimensional simplicial complex which is connected locally finite. A map f from $[a, b]$ to K is called a broken geodesic if there is a subdivision $a = t_0 < t_1 < \dots < t_{p+1} = b$ such that $f([t_i, t_{i+1}])$ is contained in some cell and the restriction of f to $[t_i, t_{i+1}]$ is a geodesic inside that cell. Then define the length of the broken geodesic map f to be:

$$L(f) = \sum_{i=0}^{i=p} d(f(t_i), f(t_{i+1})).$$

The length inside each cell being measured with respect its metric.

Then define $\tilde{d}(x, y)$, for every two points x, y in K , to be the lower bound of the lengths of broken geodesics from x to y . \tilde{d} is a pseudo-distance.

If K is connected and locally finite, then (K, \tilde{d}) is length space which is geodesic space if complete (see also [6]).

A l -simplex in K is called a *boundary simplex* if it is adjacent to exactly one $l+1$ simplex. The complex K is called *boundaryless* if there are no boundary simplices in K .

The (open) *star* of an open simplex Δ° (i.e. the topological interior of Δ or the points of Δ not belonging to any sub-face of Δ , so if Δ is point then $\Delta^\circ = \Delta$) of K is defined as:

$$st(\Delta^\circ) = \bigcup \{ \Delta_i^\circ : \Delta_i \text{ is simplex of } K \text{ with } \Delta_i \supset \Delta \}.$$

The star $st(p)$ of point p is defined as the star of its *carrier*, the unique open simplex Δ° containing p . Every star is path connected and contains the star of its points. In particular K is locally path connected. The closure of any star is sub-complex.

We say that the complex K is *admissible*, if it is dimensionally homogeneous and for every connected open subset U of K , the open set $U \setminus \{U \cap \{\text{the } (k-2) - \text{skeleton}\}\}$ is connected (k is the dimension of K) (i.e. K is $(n-1)$ -chainable).

Let $x \in K$ a vertex of K so that x is in the l -simplex Δ_l . We view Δ_l as an affine simplex in \mathbb{R}^l , that is $\Delta_l = \bigcap_{i=0}^l H_i$, where H_0, H_1, \dots, H_l are closed half spaces in general position, and we suppose that x is in the topological interior of H_0 . The Riemannian metric g_{Δ_l} is the restriction to Δ_l of a smooth Riemannian metric defined in an open neighborhood V of Δ_l in \mathbb{R}^l . The intersection $T_x \Delta_l = \bigcap_{i=1}^l H_i \subset T_x V$ is a cone with apex $0 \in T_x V$, and $g_{\Delta_l}(x)$ turns it into an euclidean cone. Let $\Delta_m \subset \Delta_l$ ($m < l$) be another simplex adjacent to x . Then, the face of $T_x \Delta_l$ corresponding to Δ_m is isomorphic to $T_x \Delta_m$ and we view $T_x \Delta_m$ as a subset of $T_x \Delta_l$.

Set $T_x K = \bigcup_{\Delta_i \ni x} T_x \Delta_i$, we call it the *tangent cone* of K at x . Let $S_x \Delta_l$ denote the subset of all unit vectors in $T_x \Delta_l$ and set $S_x = S_x K = \bigcup_{\Delta_i \ni x} S_x \Delta_i$. The set S_x is called the *link* of x in K . If Δ_l is a simplex adjacent to x , then

$g_{\Delta_l}(x)$ defines a Riemannian metric on the $(l-1)$ -simplex $S_x\Delta_l$. The family g_x of Riemannian metrics $g_{\Delta_l}(x)$ turns $S_x\Delta_l$ into a simplicial complex with a piecewise smooth Riemannian metric such that the simplices are spherical.

We call an admissible connected locally finite simplicial complex, endowed with a piecewise smooth Riemannian metric, an *admissible Riemannian complex*.

3.2 Riemannian polyhedron [9], [1].

We mean by *polyhedron* a connected locally compact separable Hausdorff space X for which there exists a simplicial complex K and homeomorphism $\theta : K \rightarrow X$. Any such pair (K, θ) is called a *triangulation* of X . The complex K is necessarily countable and locally finite (cf. [27] page 120) and the space X is path connected and locally contractible. The *dimension* of X is by definition the dimension of K and it is independent of the triangulation.

A *sub-polyhedron* of a polyhedron X with given triangulation (K, θ) , is polyhedron $X' \subset X$ having as a triangulation $(K', \theta|_{K'})$ where K' is a subcomplex of K (i.e. K' is complex whose vertices and simplexes are some of those of K).

If X is polyhedron with specified triangulation (K, θ) , we shall speak of vertices, simplexes, i -skeletons or stars of X respectively of a space of links or tangent cones of X as the image under θ of vertices, simplexes, i -skeletons or stars of K respectively the image of space of links or tangent cones of K . Thus our simplexes become compact subsets of X and the i -skeletons and stars become sub-polyhedrons of X .

If for given triangulation (K, θ) of the polyhedron X , the homeomorphism θ is locally bi-lipschitz then X is said *Lip polyhedron* and θ *Lip homeomorphism*.

A *null set* in a Lip polyhedron X is a set $Z \subset X$ such that Z meets every maximal simplex Δ , relative to a triangulation (K, θ) (hence any,) in set whose pre-image under θ has n -dimensional Lebesgue measure 0, $n = \dim \Delta$. Note that 'almost everywhere' (a.e.) means everywhere except in some null set.

A *Riemannian polyhedron* $X = (X, g)$ is defined as a Lip polyhedron X with a specified triangulation (K, θ) such that K is simplicial complex endowed with a covariant bounded measurable Riemannian metric tensor g , satisfying the ellipticity condition below. In fact, suppose that X has homogeneous dimension n and choose a measurable Riemannian metric g_Δ on the open euclidean n -simplex $\theta^{-1}(\Delta^\circ)$ of K . In terms of euclidean coordinates $\{x_1, \dots, x_n\}$ of points $x = \theta^{-1}(p)$, g_Δ thus assigns to almost every point $p \in \Delta^\circ$ (or x), an $n \times n$ symmetric positive definite matrix $g_\Delta = (g_{ij}^\Delta(x))_{i,j=1,\dots,n}$ with measurable real entries and there is a constant $\Lambda_\Delta > 0$ such that (ellipticity condition):

$$\Lambda_\Delta^{-2} \sum_{i=0}^{i=n} (\xi^i)^2 \leq \sum_{i,j} g_{ij}^\Delta(x) \xi^i \xi^j \leq \Lambda_\Delta^2 \sum_{i=0}^{i=n} (\xi^i)^2$$

for a.e. $x \in \theta^{-1}(\Delta^\circ)$ and every $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$. This condition amounts to the components of g_Δ being bounded and it is independent not only of the choice of the euclidean frame on $\theta^{-1}(\Delta^\circ)$ but also of the chosen triangulation.

For simplicity of statements we shall sometimes require that, relative to a fixed triangulation (K, θ) of Riemannian polyhedron X (uniform ellipticity condition),

$$\Lambda := \sup\{\Lambda_\Delta : \Delta \text{ is simplex of } X\} < \infty .$$

A Riemannian polyhedron X is said to be *admissible* if for a fixed triangulation (K, θ) (hence any) the Riemannian simplicial complex K is admissible.

We underline that (for simplicity) the given definition of a Riemannian polyhedron (X, g) contains already the fact (because of the definition above of the Riemannian admissible complex) that the metric g is *continuous* relative to some (hence any) triangulation (i.e. for every maximal simplex Δ the metric g_Δ is continuous up to the boundary). This fact is some times in the literature taken off. The polyhedron is said to be simplexwise smooth if relative to some triangulation (K, θ) (and hence any), the complex K is simplexwise smooth. Both continuity and simplexwise smoothness are preserved under subdivision.

In the case of a general bounded measurable Riemannian metric g on X , we often consider, in addition to g , the *euclidean Riemannian metric* g^e on the Lip polyhedron X with a specified triangulation (K, θ) . For each simplex Δ , g_Δ^e is defined in terms of euclidean frame on $\theta^{-1}(\Delta^o)$ as above by unitmatrix (δ_{ij}) . Thus g^e is by no means covariantly defined and should be regarded as a mere reference metric on the triangulated polyhedron X .

Relative to a given triangulation (K, θ) of an n -dimensional Riemannian polyhedron (X, g) (not necessarily admissible), we have on X the distance function e induced by the euclidean distance on the euclidean space V in which K is affinely Lip embedded. This distance e is not intrinsic but it will play an auxiliary role in defining an equivalent distance d_X as follows:

Let \mathfrak{Z} denote the collection of all null sets of X . For given triangulation (K, θ) consider the set $Z_K \subset \mathfrak{Z}$ obtained from X by removing from each maximal simplex Δ in X those points of Δ^o which are Lebesgue points for g_Δ . For $x, y \in X$ and any $Z \in \mathfrak{Z}$ such that $Z \subset Z_K$ we set:

$$d_X(x, y) = \sup_{\substack{Z \in \mathfrak{Z} \\ Z \supset Z_K}} \inf_{\gamma(a)=x, \gamma(b)=y} \{L_K(\gamma) : \gamma \text{ is Lip continuous path and transversal to } Z\},$$

where $L_K(\gamma)$ is the length of the path γ defined as:

$$L_K(\gamma) = \sum_{\Delta \subset X} \int_{\gamma^{-1}(\Delta^o)} \sqrt{(g_{ij}^\Delta \circ \theta^{-1} \circ \gamma) \dot{\gamma}^i \dot{\gamma}^j}, \text{ the sum is over all simplexes meeting } \gamma.$$

It is shown in [12] that the distance d_X is intrinsic, in particular it is independent of the chosen triangulation and it is equivalent to the euclidean distance e (due to the Lip affinely and homeomorphically embedding of X in some euclidean space V).

4. Energy of maps.

The concept of energy in the case of a map of Riemannian domain into an arbitrary metric space Y was defined and investigated by Korevaar and Shoen [21]. Later this concept was extended by Eells and Fuglede [12] to the case of map from an admissible Riemannian polyhedron X with simplexwise smooth Riemannian metric. Thus, The energy $E(\varphi)$ of a map φ from X to the space Y is defined as the limit of suitable approximate energy expressed in terms of the distance function d_Y of Y .

It is shown in [12] that the maps $\varphi : X \rightarrow Y$ of finite energy are precisely those quasicontinuous (i.e. has a continuous restriction to closed sets, whose complements have arbitrarily small capacity, (cf. [12] page 153) whose restriction to each top dimensional simplex of X has finite energy in the sense of Korevaar-Schoen, and $E(\varphi)$ is the sum of the energies of these restrictions.

Just now, let (X, g) be an admissible m -dimensional Riemannian polyhedron with simplexwise smooth Riemannian metric. It is not required that g is continuous across lower dimensional simplexes. The target (Y, d_Y) is an arbitrary metric space.

Denote $L_{loc}^2(X, Y)$ the space of all μ_g -mesurable (μ_g the volume measure of g) maps $\varphi : X \rightarrow Y$ having separable essential range and for which the map $d_Y(\varphi(\cdot), q) \in L_{loc}^2(X, \mu_g)$ (i.e. locally μ_g -squared integrable) for some point q (hence by triangle inequality for any point). For $\varphi, \psi \in L_{loc}^2(X, Y)$ define their distance $D(\varphi, \psi)$ by:

$$D^2(\varphi, \psi) = \int_X d_Y^2(\varphi(x), \psi(x)) d\mu_g(x).$$

Two maps $\varphi, \psi \in L_{loc}^2(X, Y)$ are said to be *equivalent* if $D(\varphi, \psi) = 0$, i.e. $\varphi(x) = \psi(x)$ μ_g -a.e. If the space X is compact then $D(\varphi, \psi) < \infty$ and D is a metric on $L_{loc}^2(X, Y) = L^2(X, Y)$ and complete if the space Y is complete [21].

The *approximate energy density* of the map $\varphi \in L_{loc}^2(X, Y)$ is defined for $\epsilon > 0$ by:

$$e_\epsilon(\varphi)(x) = \int_{B_X(x, \epsilon)} \frac{d_Y^2(\varphi(x), \varphi(x'))}{\epsilon^{m+2}} d\mu_g(x').$$

The function $e_\epsilon(\varphi) \geq 0$ is locally μ_g -integrable.

The *energy* $E(\varphi)$ of a map φ of class $L_{loc}^2(X, Y)$ is:

$$E(\varphi) = \sup_{f \in C_c(X, [0, 1])} (\limsup_{\epsilon \rightarrow 0} \int_X f e_\epsilon(\varphi) d\mu_g),$$

where $C_c(X, [0, 1])$ denotes the space of continuous functions from X to the interval $[0, 1]$ with compact support.

A map $\varphi : X \rightarrow Y$ is said *locally of finite energy*, and we write $\varphi \in W_{loc}^{1,2}(X, Y)$, if $E(\varphi|U) < \infty$ for every relatively compact domain $U \subset X$, or equivalently if X can be covered by domains $U \subset X$ such that $E(\varphi|U) < \infty$.

For example (cf. [12] lemma 4.4), every Lip continuous map $\varphi : X \rightarrow Y$ is of class $W_{loc}^{1,2}(X, Y)$. In the case when X is compact $W_{loc}^{1,2}(X, Y)$ is denoted $W^{1,2}(X, Y)$ the space of all maps of finite energy.

$W_c^{1,2}(X, Y)$ denotes the linear subspace of $W^{1,2}(X, Y)$ consisting of all maps of finite energy of compact support in X .

We denote the closure of the space $Lip_c(X)$ (the space of Lipschitz continuous functions with compact supports) in the space $W^{1,2}(X)$, $W_0^{1,2}(X)$.

We can show (cf. [12] theorem 9.1) that a map $\varphi \in L_{loc}^2(X)$ is locally of finite energy iff there is a function $e(\varphi) \in L_{loc}^1(X)$, named *energy density* of φ , such that (weak convergence):

$$\lim_{\epsilon \rightarrow 0} \int_X f e_\epsilon(\varphi) d\mu_g = \int_X f e(\varphi) d\mu_g, \text{ for each } f \in C_c(X).$$

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